

# CLUSTER ALGEBRA AND COMPLEX VOLUME OF ONCE-PUNCTURED TORUS BUNDLES AND TWO-BRIDGE KNOTS

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ABSTRACT. We propose a method to compute complex volume of 2-bridge knot complements. Clarified is a relationship between cluster variables with coefficients and canonical decompositions of knot complements.

## 1. Introduction

The cluster algebra was introduced by Fomin & Zelevinsky in [6], and it has been studied extensively since then. The characteristic operation in cluster algebra called “mutation” is related to various notions, and there exists many application of cluster algebra to representation theory of Lie algebras and quantum groups, triangulated surface [4, 5], Teichmüller theory [3], integrable systems, and so on.

As one of applications to geometry, the cluster algebraic technique is used to hyperbolic structure of fibered bundles [13], where cluster  $y$ -variables are identified with moduli of ideal hyperbolic tetrahedra. Our purpose in this paper is to study complex volume of 2-bridge knot complements via cluster variables with coefficients. The complex volume is a complexification of hyperbolic volume,

$$\mathrm{Vol}(M) + i \, \mathrm{CS}(M),$$

where  $\mathrm{Vol}(M)$  is the hyperbolic volume and  $\mathrm{CS}(M)$  is the Chern–Simons invariant of  $M$ . Based on canonical decompositions of 2-bridge knot complements in [17], we clarify a relationship between ideal tetrahedra and cluster mutations. Main observation is that the cluster variable with coefficients is closely related to Zickert’s formulation of complex volume [21], and that the complex volume is given only from the cluster variable (Theorem 4.8, also Remark 4.9). We shall also give a formula of complex volume for once-punctured torus bundle over the circle (Theorem 3.10).

There may be natural extensions of our results. One of them is a quantization of cluster algebra, which will be helpful in studies of Volume Conjecture [10, 11], a relationship between a hyperbolic geometry and quantum invariants. Indeed in the case of once-punctured torus bundle, a classical limit of adjoint action of mutations and its relationship with [1, 9] are studied in [18]. Also a generalization to higher rank [3] remains for future works.

This paper is organized as follows. In section 2, we briefly review the definition of cluster algebra and three-dimensional hyperbolic geometry, and explain their interrelationship by taking a simple example. Section 3 is devoted to once-punctured torus

bundles over the circle. We formulate hyperbolic volume via  $y$ -variables in §3.1, and complex volume via cluster variables in §3.3. Section 4 is for 2-bridge knots. First we review a canonical decomposition of 2-bridge knot complements, and we describe it in terms of cluster algebra. The key is to introduce a tropical semifield (Definition 2.4) to which the cluster coefficients belong. The hyperbolic volume via  $y$ -variables and the complex volume via cluster variables are respectively formulated in §4.2 and §4.4.

## 2. Cluster Algebra and 3-Dimensional Hyperbolic Geometry

### 2.1 Cluster Algebra

We follow a definition of cluster algebras in [6, 7]. Let  $(\mathbb{P}, \oplus, \cdot)$  be a semifield endowed an auxiliary addition  $\oplus$ , which is commutative, associative, and distributive with respect to the group multiplication  $\cdot$  in  $\mathbb{P}$ . Let  $\mathbb{QP}$  denote the quotient field of the group ring  $\mathbb{ZP}$  of  $\mathbb{P}$ . Fix  $N \in \mathbb{Z}_{>0}$ .

**Definition 2.1.** A seed is a triple  $(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$ , where

- a cluster  $\mathbf{x} = (x_1, \dots, x_N)$  is an  $N$ -tuple of  $N$  algebraically independent variables with coefficients in  $\mathbb{QP}$ ,
- a coefficient tuple  $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N)$  is an  $N$ -tuple of elements in  $\mathbb{P}$ ,
- an exchange matrix  $\mathbf{B} = (b_{ij})$  is an  $N \times N$  skew symmetric integer matrix.

We call  $x_i$  a cluster variable, and  $\varepsilon_i$  a coefficient.

**Definition 2.2.** Let  $(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$  be a seed. For each  $k = 1, \dots, N$ , we define the mutation of  $(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$  by  $\mu_k$  as

$$\mu_k(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B}) = (\tilde{\mathbf{x}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\mathbf{B}}),$$

where

- the cluster  $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_N)$  is

$$\tilde{x}_i = \begin{cases} x_i, & \text{for } i \neq k, \\ \frac{\varepsilon_k}{1 \oplus \varepsilon_k} \cdot \frac{1}{x_k} \prod_{j: b_{jk} > 0} x_j^{b_{jk}} + \frac{1}{1 \oplus \varepsilon_k} \cdot \frac{1}{x_k} \prod_{j: b_{jk} < 0} x_j^{-b_{jk}}, & \text{for } i = k, \end{cases} \quad (2.1)$$

- the coefficient tuple  $\tilde{\boldsymbol{\varepsilon}} = (\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_N)$  is

$$\tilde{\varepsilon}_i = \begin{cases} \varepsilon_k^{-1}, & \text{for } i = k, \\ \varepsilon_i \left( \frac{\varepsilon_k}{1 \oplus \varepsilon_k} \right)^{b_{ki}}, & \text{for } i \neq k, b_{ki} \geq 0, \\ \varepsilon_i (1 \oplus \varepsilon_k)^{-b_{ki}}, & \text{for } i \neq k, b_{ki} \leq 0, \end{cases} \quad (2.2)$$

- the exchange matrix  $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$  is

$$\tilde{b}_{ij} = \begin{cases} -b_{ij}, & \text{for } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2} (|b_{ik}| b_{kj} + b_{ik} |b_{kj}|), & \text{otherwise.} \end{cases} \quad (2.3)$$

Note that the resulted triplet  $(\tilde{\mathbf{x}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\mathbf{B}})$  is again a seed.

By starting from an initial seed  $(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$ , we iterate mutations and collect all obtained seeds. The cluster algebra  $\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$  is the  $\mathbb{Z}\mathbb{P}$ -subalgebra of the rational function field  $\mathbb{Q}\mathbb{P}(\mathbf{x})$  generated by all the cluster variables. In fact, in this paper we do not need the cluster algebra itself, but the seeds and the mutations. Further, we use the following:

**Proposition 2.3** ([7]). *Let  $\mathbf{y}$  be an  $N$ -tuple  $\mathbf{y} = (y_1, \dots, y_N)$ , defined by use of cluster  $\mathbf{x}$  and coefficient  $\boldsymbol{\varepsilon}$  as*

$$y_j = \varepsilon_j \prod_k x_k^{b_{kj}}. \quad (2.4)$$

Then we have a mutation,

$$\mu_k(\mathbf{y}, \mathbf{B}) = (\tilde{\mathbf{y}}, \tilde{\mathbf{B}}), \quad (2.5)$$

where

- $\tilde{\mathbf{y}} = (\tilde{y}_1, \dots, \tilde{y}_N)$  is analogous to (2.2),

$$\tilde{y}_i = \begin{cases} y_k^{-1}, & \text{for } i = k, \\ y_i \left( \frac{y_k}{1 + y_k} \right)^{b_{ki}}, & \text{for } i \neq k, b_{ki} \geq 0, \\ y_i (1 + y_k)^{-b_{ki}}, & \text{for } i \neq k, b_{ki} \leq 0, \end{cases} \quad (2.6)$$

- $\tilde{\mathbf{B}} = (\tilde{b}_{ij})$  is (2.3).

This proposition holds for arbitrary semifield  $(\mathbb{P}, \oplus, \cdot)$ . In this paper we call  $y_i$  a cluster  $y$ -variable, or a  $y$ -variable. Hereafter we use a tropical semifield [7].

**Definition 2.4.** Set  $\mathbb{P} = \{\delta^k \mid k \in \mathbb{Z}\}$ . Let  $(\mathbb{P}, \oplus, \cdot)$  be a semifield generated by a variable  $\delta$  with a multiplication  $\cdot$  and an addition  $\oplus$ ,

$$\delta^{k_1} \oplus \delta^{k_2} = \delta^{\min(k_1, k_2)}. \quad (2.7)$$

**Definition 2.5.** We define a map  $\psi, \psi : \mathbb{P} \longrightarrow \{-1, 1\}$ , given by substituting  $\delta = -1$  in elements of  $\mathbb{P}$ .

For the later use, we introduce the permutation acting on seeds.

**Definition 2.6.** For  $i, j \in \{1, \dots, N\}$  and  $i \neq j$ , let  $s_{i,j}$  be a permutation of subscripts  $i$  and  $j$  in seeds. For example permuted cluster  $s_{i,j}(\mathbf{x})$  is defined by

$$s_{i,j}(\cdots, x_i, \cdots, x_j, \cdots) = (\cdots, x_j, \cdots, x_i, \cdots).$$

Actions on  $\boldsymbol{\varepsilon}$  and  $\mathbf{B}$  are defined in the same manner. They induce an action on  $\mathbf{y}$ , and  $s_{i,j}(\mathbf{y})$  has a same form.

## 2.2 Hyperbolic Geometry

A fundamental object in three-dimensional hyperbolic geometry is an ideal hyperbolic tetrahedron  $\triangle$  in Fig. 1 [19]. The tetrahedron is parameterized by a modulus  $z \in \mathbb{C}$ , and each dihedral angle is given as in the figure. We mean  $z'$  and  $z''$  for given modulus  $z$  by

$$z' = 1 - \frac{1}{z}, \quad z'' = \frac{1}{1 - z}. \quad (2.8)$$

The cross section by the horosphere at each vertex is similar to the triangle in  $\mathbb{C}$  with vertices 0, 1, and  $z$  as in Fig. 2. In Fig. 1, we have assigned a vertex ordering following [21], which is crucial in computing the complex volume of tetrahedra modulo  $\pi^2$ .

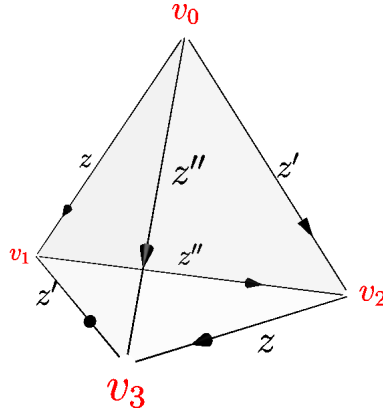


FIGURE 1. An ideal hyperbolic tetrahedron  $\triangle$  with modulus  $z$ . Dihedral angles are given by  $z$ ,  $z' = 1 - 1/z$ , and  $z'' = 1/(1 - z)$ . Each  $v_a$  denotes a vertex ordering. We give an orientation to an edge from  $v_a$  to  $v_b$  ( $a < b$ ).

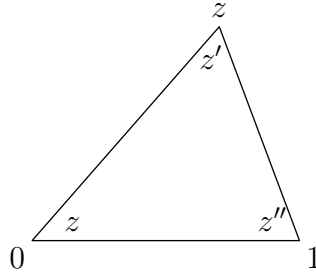


FIGURE 2. A triangle in  $\mathbb{C}$  with vertices 0, 1, and  $z$ .

The hyperbolic volume of an ideal tetrahedron  $\triangle$  with modulus  $z$  is given by the Bloch–Wigner function

$$D(z) = \Im \operatorname{Li}_2(z) + \arg(1 - z) \log |z|, \quad (2.9)$$

where  $\operatorname{Li}_2(z)$  is the dilogarithm function,

$$\operatorname{Li}_2(z) = - \int_0^z \log(1 - s) \frac{ds}{s}.$$

Note that

$$\begin{aligned} D(z) &= D(z') = D(z'') \\ &= -D(1/z) = -D(1/z') = -D(1/z''). \end{aligned} \quad (2.10)$$

See, *e.g.*, [20].

A set of ideal tetrahedra  $\{\Delta_\nu\}$  is glued together to construct a cusped hyperbolic manifold  $M$ . A modulus  $z_\nu$  of each ideal tetrahedron  $\Delta_\nu$  is determined from both gluing conditions around each edge and a completeness condition [14, 16, 19]. Then the hyperbolic volume of  $M$  is given by

$$\text{Vol}(M) = \sum_{\nu} D(z_\nu). \quad (2.11)$$

The complex volume,  $\text{Vol}(M) + i \text{CS}(M)$ , is related to an extended Rogers dilogarithm function

$$L(z; p, q) = \text{Li}_2(z) + \frac{1}{2} \log z \log(1 - z) + \frac{\pi i}{2} (q \log z + p \log(1 - z)) - \frac{\pi^2}{6}, \quad (2.12)$$

where  $p, q \in \mathbb{Z}$ . To compute the complex volume, we need an additional structure to the moduli of ideal tetrahedra:

**Definition 2.7** ([15]). A flattening of an ideal tetrahedron  $\Delta$  is

$$(w_0, w_1, w_2) = (\log z + p \pi i, -\log(1 - z) + q \pi i, \log(1 - z) - \log z - (p + q) \pi i), \quad (2.13)$$

where  $z$  is the modulus of  $\Delta$  and  $p, q \in \mathbb{Z}$ . We use  $(z; p, q)$  to denote the flattening of  $\Delta$ .

In [15], the extended pre-Bloch group is defined as the free abelian group on flattenings subject to a five-term relation, and shown is that the flattening gives the complex volume.

**Proposition 2.8** ([15]). *The complex volume of  $M$  is*

$$i (\text{Vol}(M) + i \text{CS}(M)) = \sum_{\nu} \text{sgn}(\nu) L(z_\nu; p_\nu, q_\nu), \quad (2.14)$$

where  $(z_\nu; p_\nu, q_\nu)$  and  $\text{sgn}(\nu) = \pm 1$  representing a flattening and a vertex ordering of a tetrahedron  $\Delta_\nu$ .

For a tetrahedron  $\Delta$  in Fig. 1, let  $c_{ab}$  be a complex number assigned to an edge connecting vertices  $v_a$  and  $v_b$ . Zickert clarified that the flattening  $(z; p, q)$  of  $\Delta$  is given by  $c_{ab}$  as follows.

**Proposition 2.9** ([21]). *When we have*

$$\frac{c_{03} c_{12}}{c_{02} c_{13}} = \pm z, \quad \frac{c_{01} c_{23}}{c_{03} c_{12}} = \pm \left(1 - \frac{1}{z}\right), \quad \frac{c_{02} c_{13}}{c_{01} c_{23}} = \pm \frac{1}{1 - z}, \quad (2.15)$$

the flattening  $(z; p, q)$  is given by

$$\begin{aligned} \log z + p \pi i &= \log c_{03} + \log c_{12} - \log c_{02} - \log c_{13}, \\ -\log(1 - z) + q \pi i &= \log c_{02} + \log c_{13} - \log c_{01} - \log c_{23}. \end{aligned} \quad (2.16)$$

**Remark 2.10.** In gluing tetrahedra to construct  $M$ , identical edges have the same complex numbers.

### 2.3 Interrelationship

Correspondence between the cluster algebra and the hyperbolic geometry can be seen in a simple example.<sup>1</sup> We study a triangulation of surface and its flip as in Fig. 3. Triangulation is related to quiver where the number of edges in a triangulation is the same as the fixed number  $N$  in the cluster algebra, and flip can be regarded as mutation, as depicted in the figure. Note that the exchange matrix  $\mathbf{B} = (b_{ij})$  of quiver is

$$b_{ij} = \#\{\text{arrows from } i \text{ to } j\} - \#\{\text{arrows from } j \text{ to } i\}.$$

By definition (2.4), the mutation  $\mu_3(\mathbf{y}, \mathbf{B}) = (\tilde{\mathbf{y}}, \tilde{\mathbf{B}})$ , is explicitly written as

$$\begin{aligned} \tilde{y}_1 &= y_1 (1 + y_3), \\ \tilde{y}_2 &= y_2 (1 + y_3^{-1})^{-1}, \\ \tilde{y}_3 &= y_3^{-1}, \\ \tilde{y}_4 &= y_4 (1 + y_3^{-1})^{-1}, \\ \tilde{y}_5 &= y_5 (1 + y_3). \end{aligned} \tag{2.17}$$

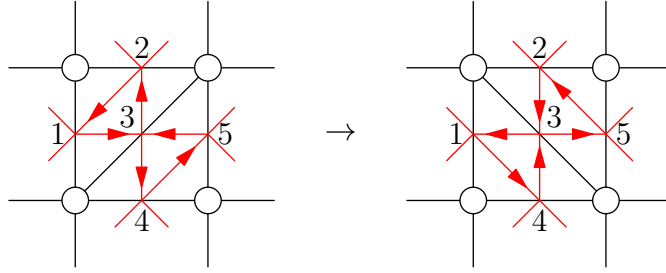


FIGURE 3. Triangulation of a punctured surface. Associated quiver is depicted in red.

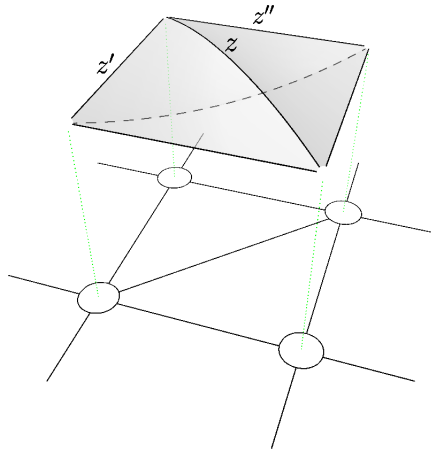


FIGURE 4. Flip and attachment of pleated tetrahedron.

On the other hand, we may regard a flip in Fig. 3 as an attachment of ideal tetrahedron  $\triangle$  with modulus  $z$  whose faces are pleated. See Fig. 4. When we denote  $z_k$  as a dihedral

<sup>1</sup>We thank T. Dimofte.

angle on edge  $k$ , dihedral angle  $\tilde{z}_k$  after attaching  $\triangle$  is given by

$$\begin{aligned}\tilde{z}_1 &= z_1 z', \\ \tilde{z}_2 &= z_2 z'', \\ \tilde{z}_3 &= z, \\ \tilde{z}_4 &= z_4 z'', \\ \tilde{z}_5 &= z_5 z',\end{aligned}\tag{2.18}$$

with a hyperbolic gluing condition

$$z_3 z = 1.$$

Comparing (2.17) with (2.18), we observe that the cluster  $y$ -variable is related to dihedral angle by

$$y_k = -z_k,$$

and especially a modulus of ideal tetrahedron  $\triangle$  is given by

$$z = -\frac{1}{y_3},\tag{2.19}$$

where a subscript “3” is a direction of mutation.

### 3. Once-Punctured Torus Bundle over $S^1$

#### 3.1 $y$ -pattern and Hyperbolic Volume

Let  $\Sigma_{1,1}$  be a once-punctured torus,  $(\mathbb{R}^2 \setminus \mathbb{Z}^2)/\mathbb{Z}^2$ . We set  $M_\varphi$  to be the once-punctured torus bundle over the circle, whose monodromy is determined by a mapping class  $\varphi \in SL(2; \mathbb{Z})$ . More precisely, via  $\varphi : \Sigma_{1,1} \rightarrow \Sigma_{1,1}$  we define identification  $(x, 0) \sim (\varphi(x), 1)$  for  $\forall x \in \Sigma_{1,1}$ , and set  $M_\varphi = \Sigma_{1,1} \times [0, 1]/\sim$ . It is known that  $M_\varphi$  is hyperbolic when  $\varphi$  has distinct real eigenvalues, and that up to conjugation we have

$$\varphi = R^{s_1} L^{t_1} \cdots R^{s_n} L^{t_n},\tag{3.1}$$

where

$$R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

To denote a mapping class (3.1) we use a sequence of symbols  $F_1 F_2 \cdots F_c = \underbrace{R \cdots R}_{s_1} \underbrace{L \cdots L}_{t_1} \cdots \underbrace{R \cdots R}_{s_n} \underbrace{L \cdots L}_{t_n}$  where  $F_k = R$  or  $L$ , and

$$c = \sum_{j=1}^n (s_j + t_j).\tag{3.2}$$

We set a triangulation of  $\Sigma_{1,1}$  as depicted in Fig. 5. It is related to the Farey triangles [2], and we will explain these Farey triangles later in the subsequent section. The actions of  $R$  and  $L$  are interpreted as “flips” of triangulation as shown in the figure. The

triangulation is translated into the cluster algebra of  $N = 3$  with the exchange matrix as

$$\mathbf{B} = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}. \quad (3.3)$$

This denotes the quiver in Fig. 6, where each vertex has a labeling corresponding to that of an edge in the triangulation. Then the flips  $R$  and  $L$  can be identified with mutations in the cluster algebra (cf. [18]), and we have

$$R = s_{1,3} \mu_1, \quad L = s_{2,3} \mu_2, \quad (3.4)$$

where  $s_{i,j}$  is a permutation defined at Definition 2.6. See Fig. 5. We have used permutations so that the exchange matrix (3.3) is invariant under these actions. In this way the flips  $R$  and  $L$  act on  $y$ -variable respectively as

$$(\mathbf{y}, \mathbf{B}) \xrightarrow{R} (R(\mathbf{y}), \mathbf{B}), \quad (\mathbf{y}, \mathbf{B}) \xrightarrow{L} (L(\mathbf{y}), \mathbf{B}), \quad (3.5)$$

where

$$R(\mathbf{y}) = \begin{pmatrix} y_3(1+y_1^{-1})^{-2} \\ y_2(1+y_1)^2 \\ y_1^{-1} \end{pmatrix}^\top, \quad L(\mathbf{y}) = \begin{pmatrix} y_1(1+y_2^{-1})^{-2} \\ y_3(1+y_2)^2 \\ y_2^{-1} \end{pmatrix}^\top. \quad (3.6)$$

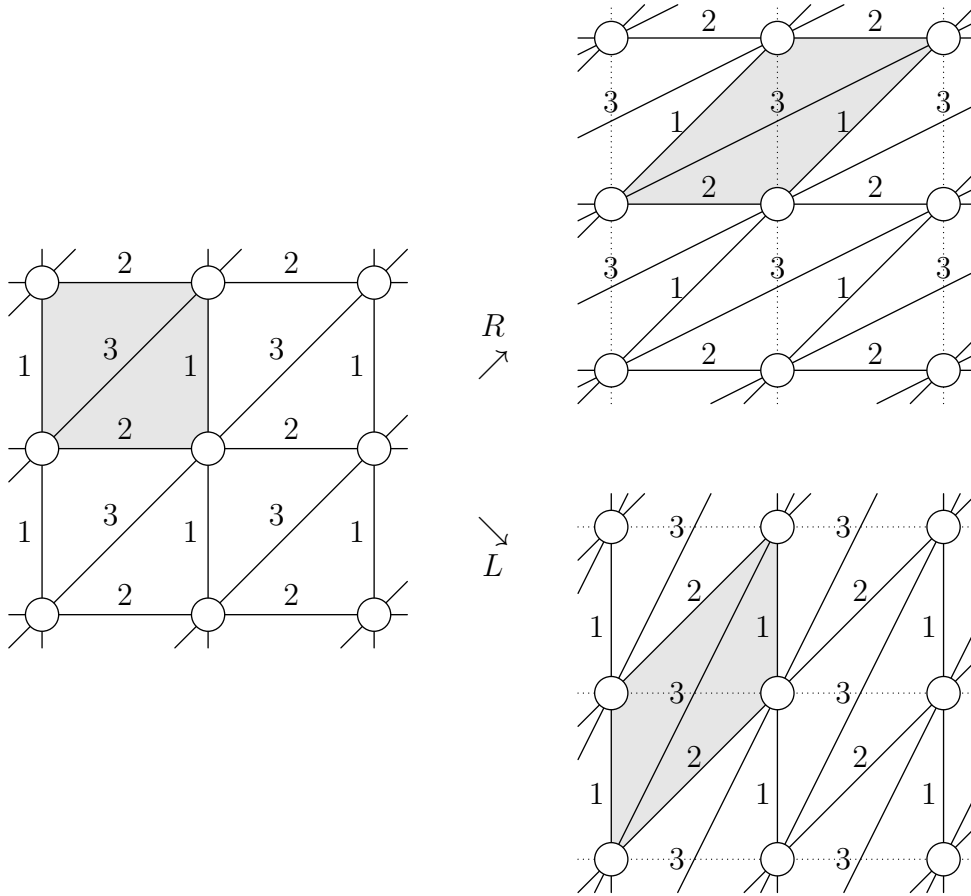


FIGURE 5. Triangulation of once-punctured torus (left). The vertex denotes a puncture. A fundamental region is colored gray. A labeling of each edge corresponds to that of each vertex in the quiver. Actions of flips,  $R$  and  $L$ , are given in the right hand side.



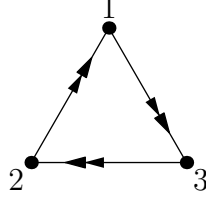


FIGURE 6. A quiver associated to a triangulation of once-puncture torus  $\Sigma_{1,1}$ .

**Definition 3.1.** A  $y$ -pattern of a mapping class  $\varphi = F_1 \cdots F_c$  (3.1) is  $\mathbf{y}[k]$  for  $k = 1, 2, \dots, c+1$  defined recursively by

$$\mathbf{y}[k+1] = F_k(\mathbf{y}[k]). \quad (3.7)$$

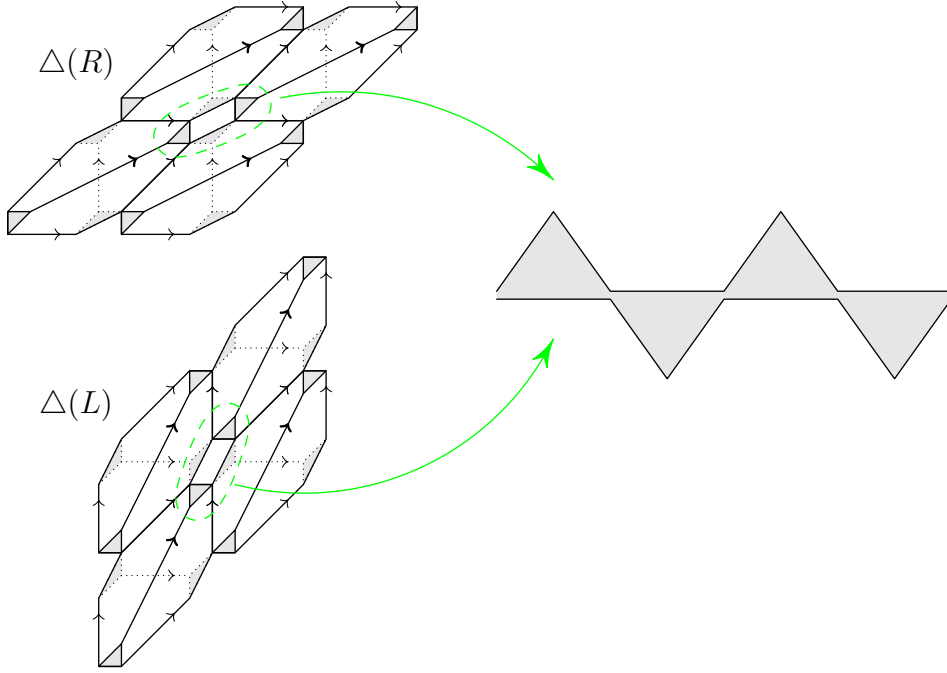


FIGURE 7. Tetrahedra  $\Delta(R)$  and  $\Delta(L)$  assigned to flip  $R$  and  $L$  on once-punctured torus. Once we fix an orientation of triangulation of  $\Sigma_{1,1}$ , an orientation of tetrahedra is induced as illustrated in the figure.

To each flip,  $R$  or  $L$ , assigned is a single ideal hyperbolic tetrahedron  $\Delta(F_k)$  as illustrated in Fig. 7, where triangulations of  $\Sigma_{1,1}$  in Fig. 5 can be regarded as the top and bottom pleated faces of ideal tetrahedron [2]. In a cross section by horosphere at a vertex we have four triangles, and each of them has one vertex not shared with any of the other three as in Fig. 7.

Our first claim is that modulus of each ideal tetrahedron is given from a  $y$ -pattern. See also [13].

**Proposition 3.2.** Let  $\mathbf{y}[k]$  be a  $y$ -pattern of  $\varphi$  with an initial condition,

$$\mathbf{y}[1] = \left( y_1, y_2, \frac{1}{y_1 y_2} \right). \quad (3.8)$$

Here  $y_1$  and  $y_2$  are geometric solutions of

$$\mathbf{y}[1] = \mathbf{y}[c+1], \quad (3.9)$$

such that each modulus  $z[k]$  defined by

$$z[k] = \begin{cases} -\frac{1}{y[k]_1}, & \text{when } F_k = R, \\ -\frac{1}{y[k]_2}, & \text{when } F_k = L, \end{cases} \quad (3.10)$$

is in the upper half plane  $\mathbb{H}$  for  $k = 1, \dots, c$ .

Then  $z[k]$  is a modulus of tetrahedron  $\Delta(F_k)$ .

*Remark 3.3.* We have discarded vertex orderings of tetrahedra in setting of moduli (3.10) here.

**Corollary 3.4.** *The hyperbolic volume of  $M_\varphi$  is given by*

$$\text{Vol}(M_\varphi) = \sum_{k=1}^c D(z[k]), \quad (3.11)$$

where  $z[k]$  is the modulus of  $\Delta(F_k)$  obtained as (3.10).

### 3.2 Proof of Proposition 3.2

We shall check that both gluing conditions and a completeness condition are fulfilled for a set of moduli (3.10). For this purpose, we recall a developing map of torus at infinity given in Fig. 8 (see, *e.g.*, [8]).

First of all, we have

$$\begin{aligned} z[3] (z''[2])^2 z[1] &= \frac{-1}{y[3]_1} \left( \frac{1}{1 + y[2]_1^{-1}} \right)^2 \frac{-1}{y[1]_1} \\ &= \frac{-1}{y[2]_3} \frac{-1}{y[1]_1} = 1. \end{aligned}$$

Here we have used  $\mathbf{y}[3] = R(\mathbf{y}[2])$  in the second equality, and the last equality follows from  $\mathbf{y}[2] = R(\mathbf{y}[1])$ . See (3.6) for the action of flip  $R$ . We see that this equality is nothing but a gluing condition for the second green circle from the bottom in Fig. 8. In the same manner, we can check

$$z[k+1] (z''[k])^2 z[k-1] = 1,$$

for  $2 \leq k \leq c-1$ , which is also corresponds to gluing conditions in the figure. With a help of periodic condition (3.9), we have

$$\begin{aligned} z[2] (z''[1])^2 z[c] &= \frac{-1}{y[2]_1} \left( \frac{1}{1 + y[1]_1^{-1}} \right)^2 \frac{-1}{y[c]_2} \\ &= 1, \end{aligned}$$

where we have used  $y[1]_3 = y[c+1]_3 = y[c]_2^{-1}$ . This coincides with a consistency condition for the right green semi-circle (top and bottom) in the figure.

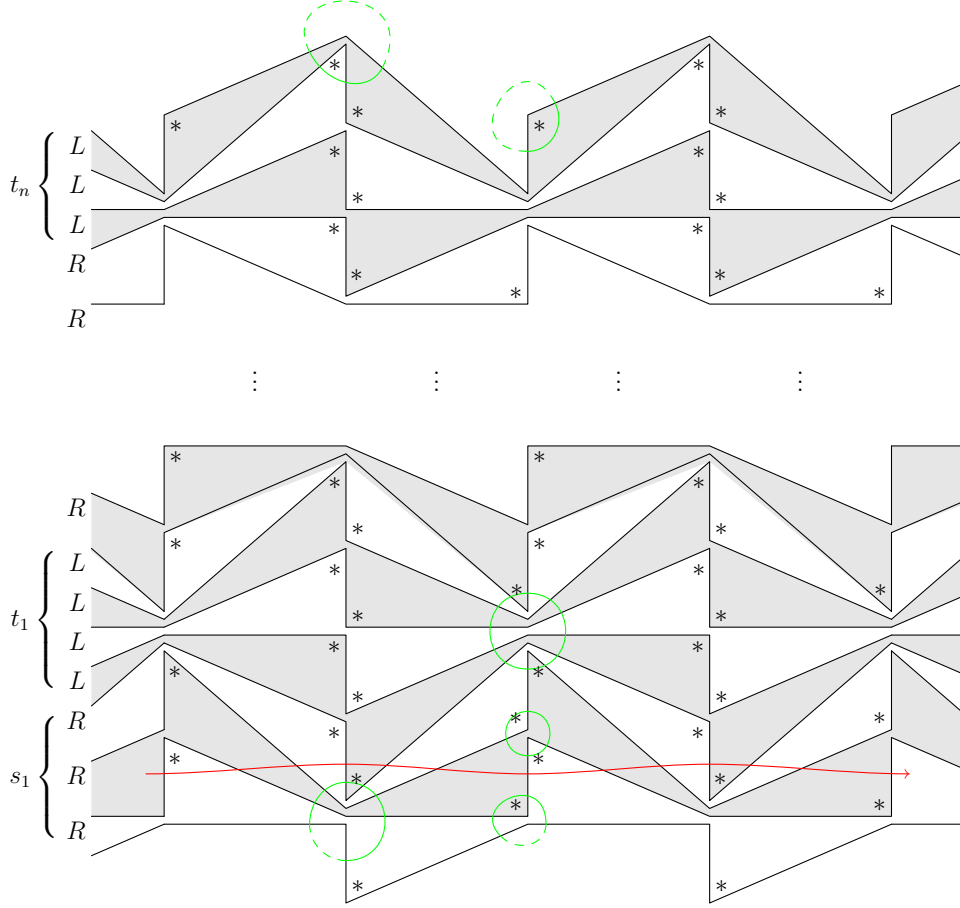


FIGURE 8. Developing map of the once-punctured torus bundle over the circle. To emphasize the layered structure, every vertex is opened up and each layer is colored alternately. Here  $*$  denotes dihedral angles  $z[k]$ , and other dihedral angles are  $z'[k]$  or  $z''[k]$  as in Fig. 2.

We can further check that

$$\begin{aligned} z[s_1 + 1] \cdot \prod_{k=1}^{s_1} (z'[k])^2 \cdot (z'[c])^2 z[c - 1] &= \frac{-1}{y[s_1 + 1]_2} \cdot \prod_{k=1}^{s_1} (1 + y[k]_1)^2 \cdot (1 + y[c]_2)^2 \frac{-1}{y[c - 1]_2} \\ &= \frac{1}{y[1]_2} y[c + 1]_2 = 1, \end{aligned}$$

due to (3.9). This identity corresponds to a gluing condition for left green semi-circle (top and bottom) in the figure. As another example of this type of equality, we have

$$z[s_1 + t_1 + 1] \cdot \prod_{k=s_1}^{s_1+t_1} (z''[k])^2 \cdot z[s_1 - 1] = 1,$$

which denotes a gluing condition for right middle large green circle. In this way, it is straightforward to check consistency conditions in the figure.

A completeness condition can be checked similarly. We have

$$\begin{aligned} \left( \frac{1}{z[1]} \frac{z'[2]}{z''[2]} \cdot \prod_{k=3}^{s_1} (z'[k])^2 \cdot z[s_1 + 1] \right)^2 &= \left( \frac{y[1]_1}{y[2]_1} \cdot \prod_{k=2}^{s_1} (1 + y[k]_1)^2 \cdot \frac{1}{y[s_1 + 1]_2} \right)^2 \\ &= \left( \frac{1}{y[1]_1 y[1]_2 y[1]_3} \right)^2 = 1, \end{aligned}$$

where the last equality follows from the initial condition (3.8). This equality is a completeness condition from a red curve in the figure.

This completes the proof.

### 3.3 Cluster Pattern and Complex Volume

We shall reformulate the preceding result by use of cluster variables with coefficients. By definition, the permuted mutations  $R$  and  $L$  (3.4) act on a seed as

$$(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B}) \xrightarrow{R} (R(\mathbf{x}, \boldsymbol{\varepsilon}), \mathbf{B}), \quad (\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B}) \xrightarrow{L} (L(\mathbf{x}, \boldsymbol{\varepsilon}), \mathbf{B}). \quad (3.12)$$

where

$$\begin{aligned} R(\mathbf{x}, \boldsymbol{\varepsilon}) &= \left( \left( \begin{array}{c} x_3 \\ x_2 \\ \frac{\varepsilon_1}{1 \oplus \varepsilon_1} \frac{x_2^2}{x_1} + \frac{1}{1 \oplus \varepsilon_1} \frac{x_3^2}{x_1} \end{array} \right)^\top, \left( \begin{array}{c} \varepsilon_3 \left( \frac{\varepsilon_1}{1 \oplus \varepsilon_1} \right)^2 \\ \varepsilon_2 (1 \oplus \varepsilon_1)^2 \\ \varepsilon_1^{-1} \end{array} \right)^\top \right), \\ L(\mathbf{x}, \boldsymbol{\varepsilon}) &= \left( \left( \begin{array}{c} x_1 \\ x_3 \\ \frac{1}{1 \oplus \varepsilon_2} \frac{x_1^2}{x_2} + \frac{\varepsilon_2}{1 \oplus \varepsilon_2} \frac{x_3^2}{x_2} \end{array} \right)^\top, \left( \begin{array}{c} \varepsilon_1 \left( \frac{\varepsilon_2}{1 \oplus \varepsilon_2} \right)^2 \\ \varepsilon_3 (1 \oplus \varepsilon_2)^2 \\ \varepsilon_2^{-1} \end{array} \right)^\top \right). \end{aligned} \quad (3.13)$$

**Definition 3.5.** A cluster pattern of  $\varphi = F_1 \cdots F_c$  (3.1) is  $(\mathbf{x}[k], \boldsymbol{\varepsilon}[k])$  for  $k = 1, 2, \dots, c+1$  defined recursively by

$$(\mathbf{x}[k+1], \boldsymbol{\varepsilon}[k+1]) = F_k(\mathbf{x}[k], \boldsymbol{\varepsilon}[k]). \quad (3.14)$$

We set an initial seed  $(\mathbf{x}, \boldsymbol{\varepsilon}, \mathbf{B})$  by

$$\mathbf{x}[1] = (x_1, x_2, x_3), \quad \boldsymbol{\varepsilon}[1] = (1, 1, 1), \quad (3.15)$$

and (3.3). Note that for all  $k$  we have

$$\boldsymbol{\varepsilon}[k] = (1, 1, 1). \quad (3.16)$$

When we set a periodic condition

$$\mathbf{x}[c+1] = \mathbf{x}[1], \quad (3.17)$$

all cluster variables  $x[k]_j$  are determined up to constant. Thanks to Prop. 2.3, the  $y$ -pattern in Prop. 3.2 can be identified with

$$\mathbf{y}[k] = \left( \left( \frac{x[k]_2}{x[k]_3} \right)^2, \left( \frac{x[k]_3}{x[k]_1} \right)^2, \left( \frac{x[k]_1}{x[k]_2} \right)^2 \right), \quad (3.18)$$

and (3.17) supports a periodicity of  $y$ -variable,  $\mathbf{y}[1] = \mathbf{y}[c+1]$ . Because of facts that a  $y$ -pattern  $\mathbf{y}[k]$  is related to moduli of  $\Delta(F_k)$ , and that cluster variable is assigned to each edge in triangulations, cluster variables can be identified with Zickert's edge parameters  $c_{ab}$  in (2.15).

*Remark 3.6.* In the present case of  $M_\varphi$ , it may not be obvious how the semifield  $\mathbb{P}$  does work. The advantage of  $\mathbb{P}$  will be clarified in the case of 2-bridge knot complements.

To get the complex volume modulo  $\pi^2$ , we need to take into account of orientation of  $\Delta(F_k)$ . We assign orientations to triangulations of  $\Sigma_{1,1}$ , and it induces an orientations of  $\Delta(F_k)$  as illustrated in Fig. 7 [15]. A tetrahedron  $\Delta(R)$  has the same orientation with one in Fig. 1, while a tetrahedron  $\Delta(L)$  has an opposite orientation. Using a relationship between the vertex ordering and dihedral angles in Fig. 1, we obtain the following.

**Lemma 3.7.** *Let  $\mathbf{x}[k]$  be a cluster pattern satisfying the condition (3.17). Then the modulus  $z[k]$  of  $\Delta(F_k)$  is given by*

$$z[k] = \begin{cases} \frac{x[k]_1 x[k+1]_3}{(x[k]_3)^2}, & \text{for } F_k = R, \\ \frac{x[k]_2 x[k+1]_3}{(x[k]_3)^2}, & \text{for } F_k = L, \end{cases} \quad (3.19)$$

for  $k = 1, \dots, c$ .

*Proof.* When  $F_k = R$ , we see from a vertex ordering of  $\Delta(R)$  in Fig. 7 and  $\Delta$  in Fig. 1 that  $-1/y[k]_1$  is identified with  $z''[k]$ . By using (3.13) and (3.18), we obtain

$$z[k] = 1 + y[k]_1 = \frac{(x[k]_2)^2 + (x[k]_3)^2}{(x[k]_3)^2} = \frac{x[k]_1 x[k+1]_3}{(x[k]_3)^2}.$$

When  $F_k = L$ , we find that  $\Delta(L)$  in Fig. 7 has an opposite vertex ordering to  $\Delta$  in Fig. 1. Thus  $(-1/y[k]_2)^{-1}$  corresponds to  $z''[k]$ , and we get

$$z[k] = 1 + \frac{1}{y[k]_2} = \frac{(x[k]_1)^2 + (x[k]_3)^2}{(x[k]_3)^2} = \frac{x[k]_2 x[k+1]_3}{(x[k]_3)^2}.$$

□

We should stress that we have discarded a vertex ordering in Prop. 3.2, and that (3.10) with (3.18) does not coincides with (3.19). See Remark 3.11 below.

*Remark 3.8.* Due to orientation of tetrahedron, a geometric solution is  $\Im(z[k]) > 0$  (resp.  $\Im(z[k]) < 0$ ) for  $F_k = R$  (resp.  $F_k = L$ ).

We obtain the flattening of  $\Delta(F_k)$  as follows.

**Lemma 3.9.** *We follow the setting at Lemma 3.7. The flattening  $(z[k]; p[k], q[k])$  for the tetrahedron  $\triangle(F_k)$  is given by*

$$\begin{aligned} \log z[k] + p[k] \pi i &= \begin{cases} \log x[k]_1 + \log x[k+1]_3 - 2 \log x[k]_3, & \text{for } F_k = R, \\ \log x[k]_2 + \log x[k+1]_3 - 2 \log x[k]_3, & \text{for } F_k = L, \end{cases} \\ -\log(1 - z[k]) + q[k] \pi i &= \begin{cases} 2 \log x[k]_3 - 2 \log x[k]_2, & \text{for } F_k = R, \\ 2 \log x[k]_3 - 2 \log x[k]_1, & \text{for } F_k = L, \end{cases} \end{aligned} \quad (3.20)$$

for  $k = 1, \dots, c$ .

*Proof.* Identification of moduli (3.19) and the actions (3.13) of flips result in

$$\frac{1}{1 - z[k]} = \begin{cases} -\left(\frac{x[k]_3}{x[k]_2}\right)^2, & \text{for } F_k = R, \\ -\left(\frac{x[k]_3}{x[k]_1}\right)^2, & \text{for } F_k = L. \end{cases} \quad (3.21)$$

From (3.19), (3.21), and Zickert's identity (2.15), the claim follows.  $\square$

The following is the main result in this section.

**Theorem 3.10.** *The complex volume of  $M_\varphi$  is*

$$i (\text{Vol}(M_\varphi) + i \text{CS}(M_\varphi)) = \sum_{k=1}^c \text{sgn}(F_k) L(z[k]; p[k], q[k]) \pmod{\pi^2}. \quad (3.22)$$

where  $\text{sgn}(R) = 1$  and  $\text{sgn}(L) = -1$ , and the flattening  $(z[k]; p[k], q[k])$  is given in (3.19) and (3.20).

*Remark 3.11.* When we discard the vertex ordering of ideal hyperbolic tetrahedra  $\triangle(F_k)$ , the resulting complex volume is defined modulo  $\frac{\pi^2}{6}$  [15]. Ignoring orientations of  $\triangle(F_k)$ , we may simply set moduli of tetrahedra from (3.10) and (3.18) as

$$z[k] = \begin{cases} -\left(\frac{x[k]_3}{x[k]_2}\right)^2, & \text{for } F_k = R, \\ -\left(\frac{x[k]_1}{x[k]_3}\right)^2, & \text{for } F_k = L, \end{cases}$$

for  $k = 1, \dots, c$ . Then we obtain the flattening  $(z[k]; p[k], q[k])$  from

$$\begin{aligned} \log z[k] + p[k] \pi i &= \begin{cases} 2 \log x[k]_3 - 2 \log x[k]_2, & \text{for } F_k = R, \\ 2 \log x[k]_1 - 2 \log x[k]_3, & \text{for } F_k = L, \end{cases} \\ -\log(1 - z[k]) + q[k] \pi i &= \begin{cases} 2 \log x[k]_2 - \log x[k]_1 - \log x[k+1]_3, & \text{for } F_k = R, \\ 2 \log x[k]_3 - \log x[k]_2 - \log x[k+1]_3, & \text{for } F_k = L. \end{cases} \end{aligned}$$

With these flattening, we have the complex volume modulo  $\frac{\pi^2}{6}$  [15] by

$$\mathrm{i} (\mathrm{Vol}(M_\varphi) + \mathrm{i} \mathrm{CS}(M_\varphi)) = \sum_{k=1}^c L(z[k]; p[k], q[k]) \mod \frac{\pi^2}{6}.$$

### 3.4 Example: $RL^2$

We take an example  $\varphi = RL^2$ . A cluster pattern is

$$\mathbf{x}[1] \xrightarrow{R} \mathbf{x}[2] \xrightarrow{L} \mathbf{x}[3] \xrightarrow{L} \mathbf{x}[4].$$

An initial cluster variable  $\mathbf{x}[1] = (x_1, x_2, x_3)$  is solved up to constant from a periodic condition (3.17),  $\mathbf{x}[1] = \mathbf{x}[4]$ :

$$x_1 = x_3, \quad \left(\frac{x_2}{x_3}\right)^4 + \left(\frac{x_2}{x_3}\right)^2 + 2 = 0.$$

By setting  $x_1 = x_3 = 1$ , geometric solutions are  $x_2 = \pm(0.6760 \cdots + \mathrm{i} \cdot 0.9783 \cdots)$ . From (3.20) the plus sign solution gives the flattening parameters  $(p[k], q[k])$  for  $k = 1, 2, 3$  as  $(0, -1)$ ,  $(0, 1)$ ,  $(0, 1)$ , while the minus sign one gives  $(0, 1)$ ,  $(-2, 1)$ ,  $(2, -1)$ . Both solutions give

$$\mathrm{Vol}(M_{RL^2}) + \mathrm{i} \mathrm{CS}(M_{RL^2}) = 2.6667 \cdots - \mathrm{i} \cdot 0.4112 \cdots.$$

## 4. 2-Bridge Knots

### 4.1 Canonical Decomposition of 2-Bridge Knot Complements

We briefly describe the canonical decomposition of a hyperbolic 2-bridge knot following [17]. See also [8]. Let  $K_{q/p}$  be a hyperbolic 2-bridge knot (or link). Here we assume that  $p$  and  $q$  are coprime integers such that  $2 \leq q < p/2$  (see, *e.g.*, [12]). When  $p$  is even,  $K_{q/p}$  is link. We use a continued fraction expression of  $q/p$ ,

$$q/p = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = [a_1, a_2, \dots, a_n], \quad (4.1)$$

where  $n \geq 1$ ,  $a_j \in \mathbb{Z}_{>0}$ , and  $a_n \geq 2$ . We set

$$c = \sum_{i=1}^n a_i. \quad (4.2)$$

Correspondingly we have the chain of Farey triangles,  $(\sigma[1], \sigma[2], \dots, \sigma[c])$  in Fig. 9. Vertices of each Farey triangle are rational numbers,  $a/b$ ,  $c/d$ , and  $(a+c)/(b+d)$ .

Each Farey triangle determines a triangulation of 4-punctured sphere  $\Sigma_{0,4} = (\mathbb{R}^2 \setminus \mathbb{Z}^2)/\Gamma$ , where  $\Gamma$  is a transformation group generated by  $\pi$ -rotations about every point in  $\mathbb{Z}^2$ . When two Farey triangles  $\sigma[k]$  and  $\sigma[k+1]$  are adjacent, a flip connects triangulations of  $\Sigma_{0,4}$  as in Fig. 10. Namely when  $\sigma[k+1]$  is in the right (resp. left) to  $\sigma[k]$ , a flip  $R$  (resp.  $L$ ) acts on triangulation of  $\Sigma_{0,4}$ . In constructing a canonical decomposition of

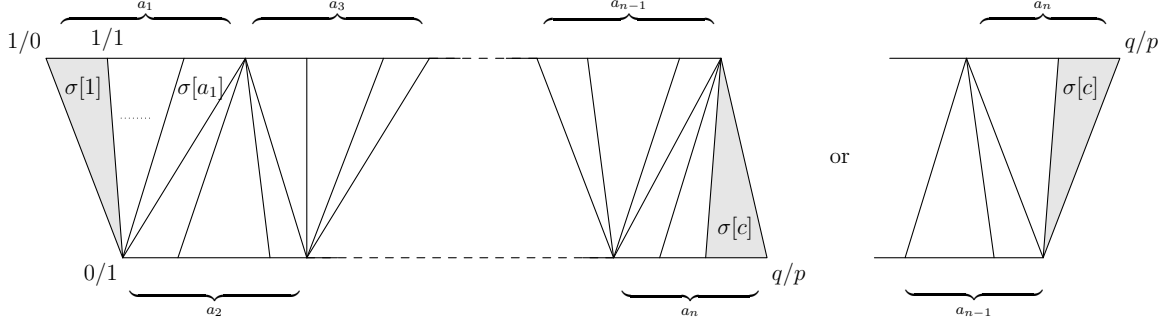


FIGURE 9. The Farey triangle

$S^3 \setminus K_{q/p}$ , triangulations for the first Farey triangle  $\sigma[1]$  is collapsed into a single edge, and the Farey triangle  $\sigma[2]$  is folded along the edge of  $1/2$ . Similarly the triangle  $\sigma[c]$  collapsed into an edge, and  $\sigma[c-1]$  is folded along an edge of  $[a_1, \dots, a_n-2]$ . We use a sequence of symbols to denote flips as

$$F_1 F_2 \cdots F_{c-3} = \begin{cases} R^{a_1-1} L^{a_2} R^{a_3} \cdots R^{a_{n-1}} L^{a_n-2}, & \text{when } n \text{ is even,} \\ R^{a_1-1} L^{a_2} R^{a_3} \cdots L^{a_{n-1}} R^{a_n-2}, & \text{when } n \text{ is odd.} \end{cases} \quad (4.3)$$

Here  $F_k$  denotes a symbol,  $F_k = R$  or  $L$ , and acts on the triangulation for  $\sigma[k+1]$ .

As in the case of the once-punctured torus bundle, we can assign ideal hyperbolic tetrahedra to flips. In the case of four-punctured sphere, a pair of ideal hyperbolic tetrahedra,  $\Delta_1(F_k)$  and  $\Delta_2(F_k)$ , are associated to the flip  $F_k$  as in Fig. 11. Here we have illustrated only a single peripheral annulus, since the combinatorics of four peripheral annuli are same. Due to folding of  $\sigma[2]$ , the first pair of tetrahedra  $\Delta_i(F_1)$  is folded at edge corresponding to  $1/2$ . Similarly the last pair of tetrahedra  $\Delta_i(F_{c-3})$  is also folded at edge corresponding to  $[a_1, \dots, a_n-2]$ .

## 4.2 $y$ -pattern and Hyperbolic Volume

We first formulate the flips (4.3) by use of mutations in the  $y$ -variables. We set a triangulation of four-punctured sphere  $\Sigma_{0,4}$  as in Fig. 10. It induces the cluster algebra of  $N = 6$  with the exchange matrix  $\mathbf{B}$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad (4.4)$$

whose quiver is in Fig. 12.

The flips,  $R$  and  $L$ , in Fig. 10 are simply identified with permuted mutations as

$$R = s_{5,6} s_{1,5} s_{2,6} \mu_1 \mu_2, \quad L = s_{5,6} s_{3,5} s_{4,6} \mu_3 \mu_4. \quad (4.5)$$

Here, as in the case of the once-punctured torus, we have inserted permutations  $s_{i,j}$  so that the exchange matrix  $\mathbf{B}$  (4.4) is invariant under  $R$  and  $L$ . Explicit actions on



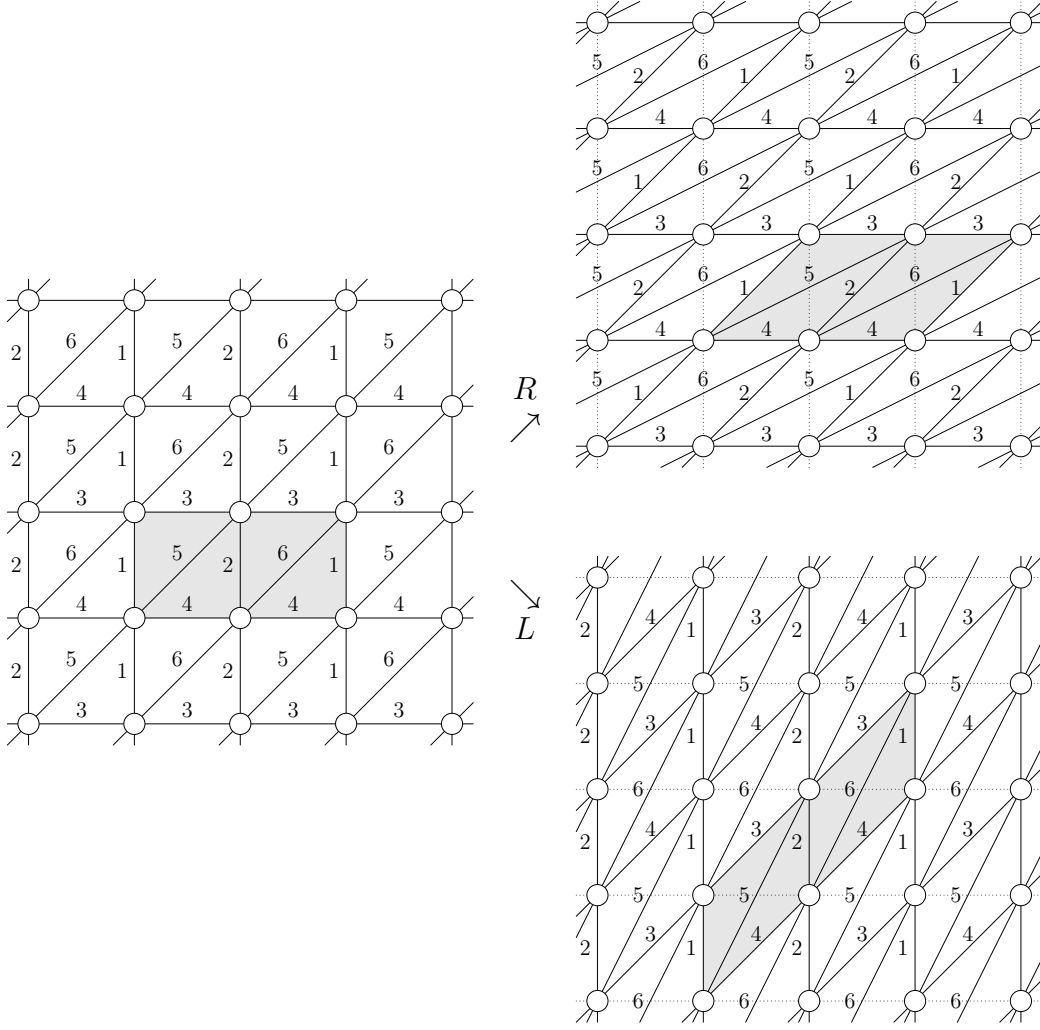


FIGURE 10. Triangulation of 4-punctured sphere  $\Sigma_{0,4}$  (left), where a fundamental region is colored gray. The flips of triangulation are given in the right hand side.

$y$ -variables are written as

$$(\mathbf{y}, \mathbf{B}) \xrightarrow{R} (R(\mathbf{y}), \mathbf{B}), \quad (\mathbf{y}, \mathbf{B}) \xrightarrow{L} (L(\mathbf{y}), \mathbf{B}), \quad (4.6)$$

where

$$R(\mathbf{y}) = \begin{pmatrix} y_5 (1 + y_1^{-1})^{-1} (1 + y_2^{-1})^{-1} \\ y_6 (1 + y_1^{-1})^{-1} (1 + y_2^{-1})^{-1} \\ y_3 (1 + y_1) (1 + y_2) \\ y_4 (1 + y_1) (1 + y_2) \\ y_2^{-1} \\ y_1^{-1} \end{pmatrix}^{\top}, \quad L(\mathbf{y}) = \begin{pmatrix} y_1 (1 + y_3^{-1})^{-1} (1 + y_4^{-1})^{-1} \\ y_2 (1 + y_3^{-1})^{-1} (1 + y_4^{-1})^{-1} \\ y_5 (1 + y_3) (1 + y_4) \\ y_6 (1 + y_3) (1 + y_4) \\ y_4^{-1} \\ y_3^{-1} \end{pmatrix}^{\top}. \quad (4.7)$$

**Definition 4.1.** A  $y$ -pattern for  $K_{q/p}$  is  $\mathbf{y}[k]$  for  $k = 1, 2, \dots, c - 2$  defined by

$$\mathbf{y}[k+1] = F_k(\mathbf{y}[k]). \quad (4.8)$$

where  $F_k$  is  $R$  or  $L$  as in (4.3).

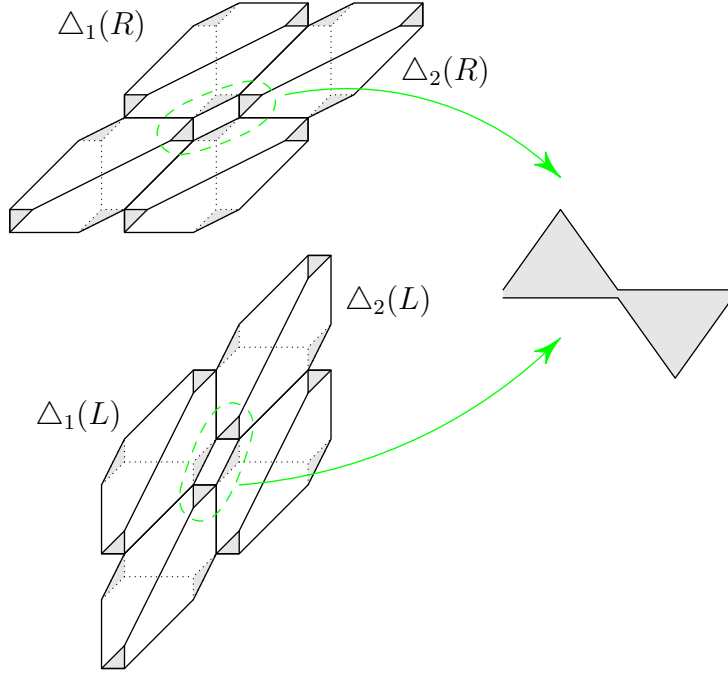


FIGURE 11. Tetrahedron  $\Delta_i(R)$  and  $\Delta_i(L)$  assigned to flip  $R$  and  $L$  on four-punctured sphere.

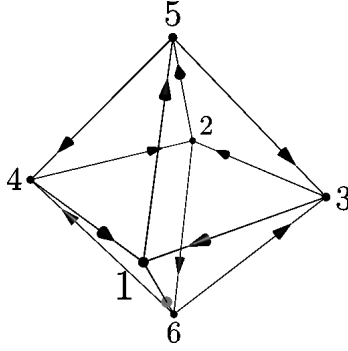


FIGURE 12. A quiver associated to triangulation of four-punctured sphere  $\Sigma_{0,4}$ .

**Proposition 4.2.** Let  $\mathbf{y}[k]$  be a  $y$ -pattern for  $K_{q/p}$  constructed from an initial condition

$$\mathbf{y}[1] = \left( y, y, -\frac{1}{y}, -\frac{1}{y}, -1, -1 \right). \quad (4.9)$$

Here  $y$  is a geometric solution of

$$\begin{cases} y[c-2]_3 = y[c-2]_4 = -1, & \text{if } n \text{ is even,} \\ y[c-2]_1 = y[c-2]_2 = -1, & \text{if } n \text{ is odd,} \end{cases} \quad (4.10)$$

such that each modulus  $z_i[k]$  for  $i = 1, 2$  and  $k = 1, 2, \dots, c-3$  defined by

$$z_i[k] = \begin{cases} -\frac{1}{y[k]_i}, & \text{if } F_k = R, \\ -\frac{1}{y[k]_{2+i}}, & \text{if } F_k = L, \end{cases} \quad (4.11)$$

is in the upper half plane  $\mathbb{H}$ .

Then  $z_i[k]$  denotes a modulus of tetrahedron  $\triangle_i(F_k)$ .

**Corollary 4.3.** *The hyperbolic volume of the knot complement  $S^3 \setminus K_{q/p}$  is given by*

$$\text{Vol}(S^3 \setminus K_{q/p}) = \sum_{k=1}^{c-3} \sum_{i=1,2} D(z_i[k]), \quad (4.12)$$

where  $z_i[k]$ 's are the moduli (4.11).

*Remark 4.4.* Under the initial condition (4.9), we have invariants of  $y$ -variable under a mutation,

$$y[k]_{i_1} y[k]_{i_2} y[k]_{i_3} = 1, \quad (4.13)$$

for  $k = 1, 2, \dots, c-2$  and  $(i_1, i_2, i_3) \in \{1, 2\} \times \{3, 4\} \times \{5, 6\}$ .

### 4.3 Proof of Proposition 4.2

We study four cases separately,

(I) even  $n$ , and  $a_n > 2$ ,

$$F_1 \cdots F_{c-3} = R^{a_1-1} L^{a_2} \cdots L^{a_n-2}, \quad y[c-2]_3 = y[c-2]_4 = -1,$$

(II) even  $n$ , and  $a_n = 2$ ,

$$F_1 \cdots F_{c-3} = R^{a_1-1} L^{a_2} \cdots R^{a_n-1}, \quad y[c-2]_3 = y[c-2]_4 = -1,$$

(III) odd  $n$ , and  $a_n > 2$ ,

$$F_1 \cdots F_{c-3} = R^{a_1-1} L^{a_2} \cdots R^{a_n-2}, \quad y[c-2]_1 = y[c-2]_2 = -1,$$

(IV) odd  $n$ , and  $a_n = 2$ ,

$$F_1 \cdots F_{c-3} = R^{a_1-1} L^{a_2} \cdots L^{a_n-1}, \quad y[c-2]_1 = y[c-2]_2 = -1.$$

We start a proof for the case (I). We recall that a  $y$ -pattern is

$$\mathbf{y}[1] \xrightarrow{R} \cdots \xrightarrow{R} \mathbf{y}[a_1] \xrightarrow{L} \mathbf{y}[a_1+1] \xrightarrow{L} \cdots \xrightarrow{L} \mathbf{y}[a_1+a_2] \xrightarrow{R} \cdots$$

A typical developing map of one peripheral annulus is depicted in Fig. 13. To see that the  $y$ -variable is related to modulus of ideal tetrahedra  $\triangle_i(F_k)$  through (4.11), we need to check that both gluing conditions and a completeness condition are fulfilled. For example, a gluing condition for a green circle in the bottom of Fig. 13 can be checked as follows;

$$\begin{aligned} z_2[2] z_1''[1] z_2''[1] &= \frac{-1}{y[2]_2} \frac{1}{1+y[1]_1^{-1}} \frac{1}{1+y[1]_2^{-1}} \\ &= \frac{-1}{y[1]_6} = 1, \end{aligned}$$

where we have used  $\mathbf{y}[2] = R(\mathbf{y}[1])$  as given in (4.7), and the last equality is from (4.9). In the same manner, a gluing condition for the second green circle from the bottom can be checked by use of  $\mathbf{y}[3] = R(\mathbf{y}[2])$  and  $\mathbf{y}[2] = R(\mathbf{y}[1])$  as

$$\begin{aligned} z_1[3] z_1''[2] z_2''[2] z_2[1] &= \frac{1}{y[3]_1} \frac{1}{1+y[2]_1^{-1}} \frac{1}{1+y[2]_2^{-1}} \frac{1}{y[1]_2} \\ &= \frac{1}{y[2]_5 y[1]_2} = 1. \end{aligned}$$

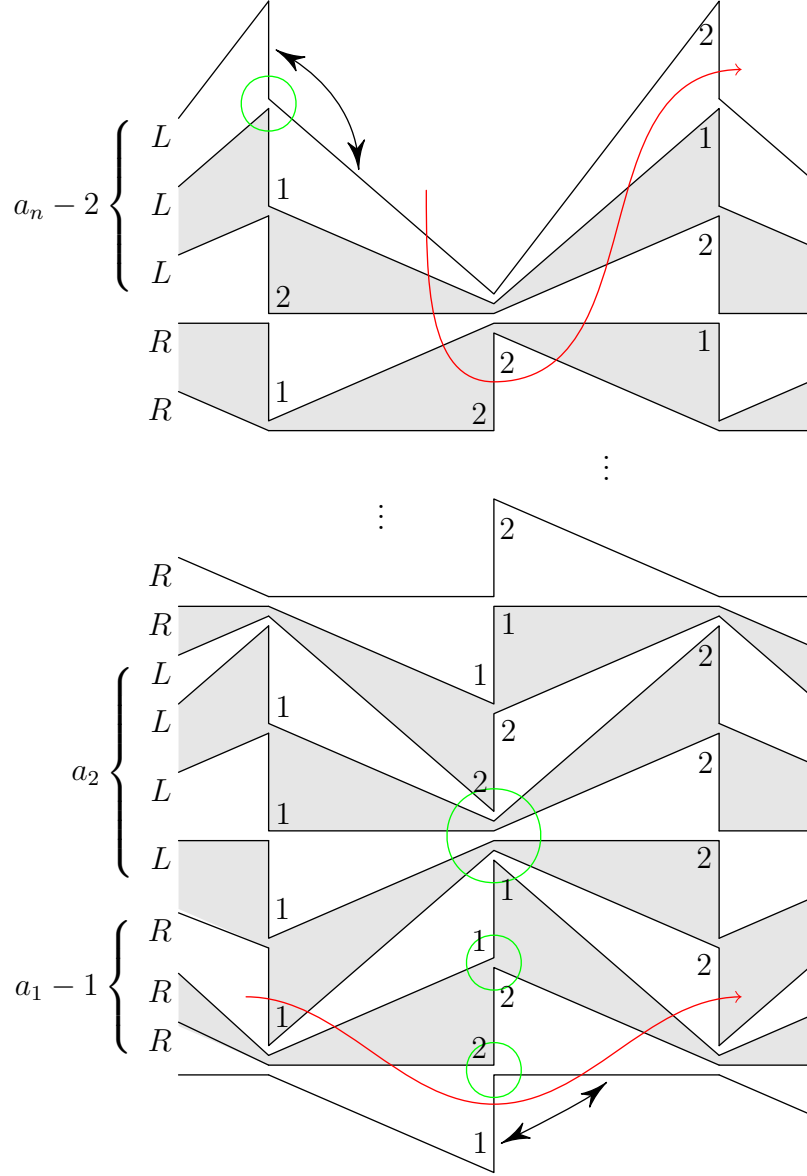


FIGURE 13. Developing map of one peripheral annulus for the case (I). To emphasize the layered structure, each vertex is opened up and each layer is colored alternately. Here 1 and 2 respectively denote dihedral angles  $z_1[k]$  and  $z_2[k]$ . Two black arrows denote an identification of edges, which are consequence of folding of  $\sigma[2]$  and  $\sigma[c-1]$ .

We can check that  $z_1[k+1] z_1''[k] z_2''[k] z_2[k-1] = 1$ , and that this type of identity denotes a gluing condition composed from four dihedral angles in Fig. 13. A gluing condition for

a green circle in the middle of the figure is also checked as follows,

$$\begin{aligned}
& z_2[a_1 + a_2] \cdot \prod_{k=a_1-1}^{a_1+a_2-1} z_1''[k] z_2''[k] \cdot z_1[a_1 - 2] \\
&= \frac{1}{y[a_1 + a_2]_2} \cdot \prod_{k=a_1}^{a_1+a_2-1} \frac{1}{1 + y[k]_3^{-1}} \frac{1}{1 + y[k]_4^{-1}} \cdot \frac{1}{1 + y[a_1 - 1]_1^{-1}} \frac{1}{1 + y[a_1 - 1]_2^{-1}} \cdot \frac{1}{y[a_1 - 2]_1} \\
&= \frac{1}{y[a_1 + a_2]_2} \cdot \prod_{k=a_1}^{a_1+a_2-1} \frac{1}{1 + y[k]_3^{-1}} \frac{1}{1 + y[k]_4^{-1}} \cdot y[a_1]_2 \\
&= \frac{1}{y[a_1 + a_2]_2} y[a_1 + a_2]_2 = 1.
\end{aligned}$$

Other gluing conditions are essentially same with above computations.

A completeness condition can be read from red curves in the figure. From the lower curve we have

$$\begin{aligned}
z_1[a_1] \cdot \prod_{k=2}^{a_1-1} z_1'[k] z_2'[k] \cdot \frac{z_2'[1]}{z_1''[1]} &= \frac{-1}{y[a_1]_3} \cdot \prod_{k=2}^{a_1-1} (1 + y[k]_1) (1 + y[k]_2) \cdot \frac{1}{y[1]_1} \\
&= \frac{-1}{y[1]_1 y[1]_3} = 1,
\end{aligned}$$

where we have used (4.9) in the last equality. It is noted that, corresponding to an upper red curve, we can check that

$$\begin{aligned}
& \frac{z_1''[c-3]}{z_2'[c-3]} \cdot \prod_{k=c-a_n-1}^{c-4} z_1''[k] z_2''[k] \cdot z_2[c - a_n - 2] \\
&= -y[c-2]_2 y[c-2]_5 = 1,
\end{aligned}$$

where (4.10) and (4.13) are used in the last equality.

This completes a proof for the case (I).

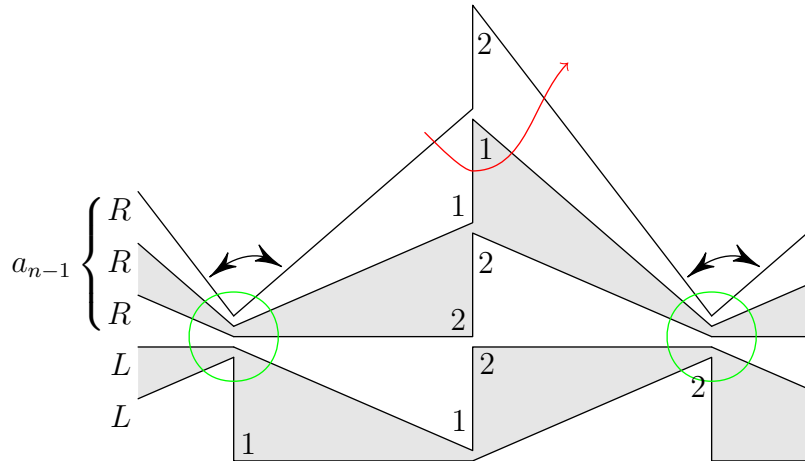


FIGURE 14. Upper part of developing map for the case (II).

In the case (II), the developing map is slightly different from the case (I): the lower part of Fig. 13 is same, but an upper part is replaced with Fig. 14. A consistency check

of gluing equations is same with the case (I). For example, a gluing equation for a green circle in Fig. 14 is

$$\begin{aligned}
& \prod_{k=c-a_{n-1}-1}^{c-3} z'_1[k] z'_2[k] \cdot z_2[c-a_{n-1}-2] \\
&= \prod_{k=c-a_{n-1}}^{c-3} (1+y[k]_1) (1+y[k]_2) \cdot (1+y[c-a_{n-1}-1]_3) (1+y[c-a_{n-1}-1]_4) \frac{-1}{y[c-a_{n-1}-2]_4} \\
&= -y[c-2]_3 = 1,
\end{aligned}$$

where the folding condition (4.10) is used in the last equality. A red curve in Fig. 14 reads as

$$\frac{z''_1[c-3]}{z'_2[c-3]} z_1[c-4] = -y[c-2]_5 y[c-2]_2 = 1,$$

where the last equality is a consequence of (4.10) and (4.13), and this denotes a completeness condition. This completes a proof for the case (II).

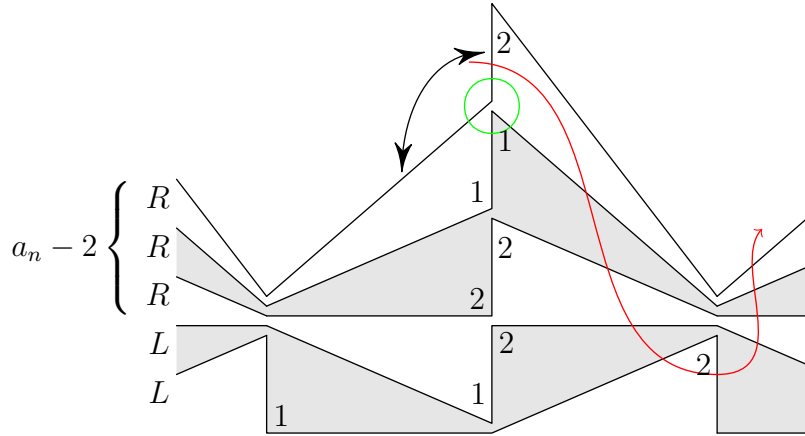


FIGURE 15. Upper part of developing map for case (III).

In the case (III), an upper part of Fig. 13 is replaced with Fig. 15. A gluing condition for a green circle in Fig. 15 can be checked as

$$\begin{aligned}
z''_1[c-3] z''_2[c-3] z_1[c-4] &= (1+y[c-3]_1^{-1})^{-1} (1+y[c-3]_2^{-1})^{-1} \frac{-1}{y[c-4]_1} \\
&= -y[c-2]_2 = 1.
\end{aligned}$$

Correspondingly a completeness condition is read from a red curve in Fig. 15, and we can check

$$\frac{z'_1[c-3]}{z''_2[c-3]} \cdot \prod_{k=c-a_n-1}^{c-4} z'_1[k] z'_2[k] \cdot z_1[c-a_n-2] = -y[c-2]_6 y[c-2]_4 = 1,$$

by using (4.10) and (4.13) at the last equality. This completes a proof for the case (III).

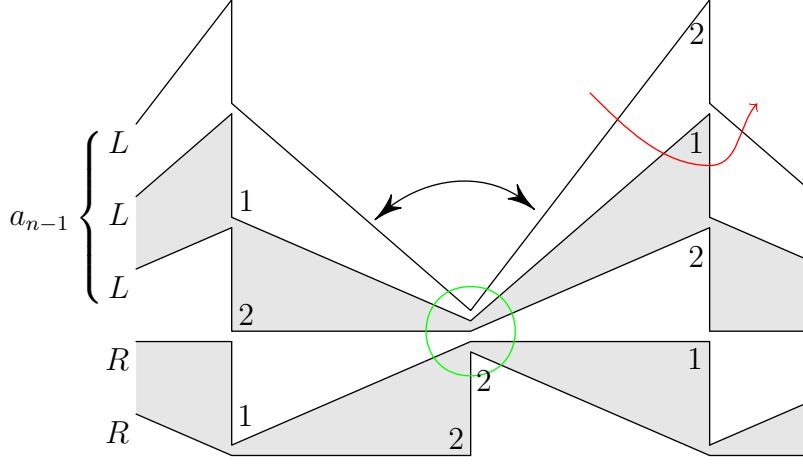


FIGURE 16. Upper part of developing map for case (IV).

In the last case (IV), an upper part of Fig. 13 is replaced by Fig. 16. A gluing condition for a green curve in Fig. 16 is checked as

$$\begin{aligned}
& \prod_{k=c-a_{n-1}-1}^{c-3} z_1''[k] z_2''[k] \cdot z_2[c - a_{n-1} - 2] \\
&= \prod_{k=c-a_{n-1}}^{c-3} \frac{1}{1 + y[k]_3^{-1}} \frac{1}{1 + y[k]_4^{-1}} \cdot \frac{1}{1 + y[c - a_{n-1} - 1]_1^{-1}} \frac{1}{1 + y[c - a_{n-1} - 1]_2^{-1}} \frac{-1}{y[c - a_{n-1} - 2]_2} \\
&= \prod_{k=c-a_{n-1}}^{c-3} (1 + y[k]_3^{-1})^{-1} (1 + y[k]_4^{-1})^{-1} \cdot (-y[c - a_{n-1}]_1) \\
&= -y[c - 2]_1 = 1.
\end{aligned}$$

Correspondingly we have a completeness condition corresponding to a red curve,

$$\begin{aligned}
\frac{z_1'[c - 3]}{z_2''[c - 3]} z_2[c - 4] &= \frac{1 + y[c - 3]_3}{y[c - 3]_3} (1 + y[c - 3]_4) \frac{-1}{y[c - 4]_4} \\
&= -y[c - 2]_3 y[c - 2]_6 = 1,
\end{aligned}$$

where we have used (4.10) and (4.13) at the last equality. This completes a proof for the case (IV).

#### 4.4 Cluster Pattern and Complex Volume

We reformulate the preceding result by use of a cluster  $\mathbf{x}$  and a coefficient tuple  $\boldsymbol{\varepsilon}$ . Actions of the two flips  $R$  and  $L$  defined as permuted mutations (4.5) are explicitly given by

$$(\mathbf{x}, \boldsymbol{\varepsilon}) \xrightarrow{R} R(\mathbf{x}, \boldsymbol{\varepsilon}), \quad (\mathbf{x}, \boldsymbol{\varepsilon}) \xrightarrow{L} L(\mathbf{x}, \boldsymbol{\varepsilon}), \quad (4.14)$$

where

$$\begin{aligned}
 R(\mathbf{x}, \boldsymbol{\varepsilon}) &= \left( \left( \begin{array}{c} x_5 \\ x_6 \\ x_3 \\ x_4 \\ \frac{\varepsilon_2}{1 \oplus \varepsilon_2} \frac{x_3 x_4}{x_2} + \frac{1}{1 \oplus \varepsilon_2} \frac{x_5 x_6}{x_2} \\ \frac{\varepsilon_1}{1 \oplus \varepsilon_1} \frac{x_3 x_4}{x_1} + \frac{1}{1 \oplus \varepsilon_1} \frac{x_5 x_6}{x_1} \end{array} \right)^\top, \left( \begin{array}{c} \varepsilon_5 \frac{\varepsilon_1}{1 \oplus \varepsilon_1} \frac{\varepsilon_2}{1 \oplus \varepsilon_2} \\ \varepsilon_6 \frac{\varepsilon_1}{1 \oplus \varepsilon_1} \frac{\varepsilon_2}{1 \oplus \varepsilon_2} \\ \varepsilon_3 (1 \oplus \varepsilon_1) (1 \oplus \varepsilon_2) \\ \varepsilon_4 (1 \oplus \varepsilon_1) (1 \oplus \varepsilon_2) \\ \varepsilon_2^{-1} \\ \varepsilon_1^{-1} \end{array} \right)^\top \right), \\
 L(\mathbf{x}, \boldsymbol{\varepsilon}) &= \left( \left( \begin{array}{c} x_1 \\ x_2 \\ x_5 \\ x_6 \\ \frac{1}{1 \oplus \varepsilon_4} \frac{x_1 x_2}{x_4} + \frac{\varepsilon_4}{1 \oplus \varepsilon_4} \frac{x_5 x_6}{x_4} \\ \frac{1}{1 \oplus \varepsilon_3} \frac{x_1 x_2}{x_3} + \frac{\varepsilon_3}{1 \oplus \varepsilon_3} \frac{x_5 x_6}{x_3} \end{array} \right)^\top, \left( \begin{array}{c} \varepsilon_1 \frac{\varepsilon_3}{1 \oplus \varepsilon_3} \frac{\varepsilon_4}{1 \oplus \varepsilon_4} \\ \varepsilon_2 \frac{\varepsilon_3}{1 \oplus \varepsilon_3} \frac{\varepsilon_4}{1 \oplus \varepsilon_4} \\ \varepsilon_5 (1 \oplus \varepsilon_3) (1 \oplus \varepsilon_4) \\ \varepsilon_6 (1 \oplus \varepsilon_3) (1 \oplus \varepsilon_4) \\ \varepsilon_4^{-1} \\ \varepsilon_3^{-1} \end{array} \right)^\top \right). \tag{4.15}
 \end{aligned}$$

**Definition 4.5.** A cluster pattern of  $K_{q/p}$  is  $(\mathbf{x}[k], \boldsymbol{\varepsilon}[k])$  for  $k = 1, 2, \dots, c-2$  defined recursively by

$$(\mathbf{x}[k+1], \boldsymbol{\varepsilon}[k+1]) = F_k(\mathbf{x}[k], \boldsymbol{\varepsilon}[k]), \tag{4.16}$$

where  $F_k$  is  $R$  or  $L$  as (4.3).

Prop. 2.3 shows that the  $y$ -pattern  $\mathbf{y}[k]$  in Prop. 4.2 can be given in terms of the cluster pattern  $(\mathbf{x}[k], \boldsymbol{\varepsilon}[k])$  as (2.4),

$$\begin{aligned}
 \mathbf{y}[k] &= \left( \varepsilon[k]_1 \frac{x[k]_3 x[k]_4}{x[k]_5 x[k]_6}, \varepsilon[k]_2 \frac{x[k]_3 x[k]_4}{x[k]_5 x[k]_6}, \varepsilon[k]_3 \frac{x[k]_5 x[k]_6}{x[k]_1 x[k]_2}, \right. \\
 &\quad \left. \varepsilon[k]_4 \frac{x[k]_5 x[k]_6}{x[k]_1 x[k]_2}, \varepsilon[k]_5 \frac{x[k]_1 x[k]_2}{x[k]_3 x[k]_4}, \varepsilon[k]_6 \frac{x[k]_1 x[k]_2}{x[k]_3 x[k]_4} \right). \tag{4.17}
 \end{aligned}$$

Folding conditions (4.9) and (4.10), which comes from folding of the Farey triangles  $\sigma[2]$  and  $\sigma[c-1]$ , are fulfilled by constraints on cluster variables and coefficients,

$$\psi(\boldsymbol{\varepsilon}[1]) = (-1, -1, 1, 1, -1, -1), \quad \text{or} \quad (1, 1, -1, -1, -1, -1), \tag{4.18}$$

$$\mathbf{x}[1] = (x, x, x, x, x_5, x_6), \tag{4.19}$$

and

$$\begin{cases} \psi(\varepsilon[c-2]_3) = \psi(\varepsilon[c-2]_4) = -1, & \text{if } n \text{ is even,} \\ \psi(\varepsilon[c-2]_2) = \psi(\varepsilon[c-2]_1) = -1, & \text{if } n \text{ is odd,} \end{cases} \tag{4.20}$$

$$\begin{cases} x[c-2]_1 = x[c-2]_2 = x[c-2]_5 = x[c-2]_6, & \text{if } n \text{ is even,} \\ x[c-2]_3 = x[c-2]_4 = x[c-2]_5 = x[c-2]_6, & \text{if } n \text{ is odd.} \end{cases} \tag{4.21}$$

Here  $\psi$  is defined in Def. 2.5. At (4.21) and in the rest of this section, we regard that the coefficients are in the image of  $\psi$  for simplicity. We should note that one of the



two conditions (4.18) is chosen so that it is consistent with the constraint (4.20). To fulfill (4.18), we set

$$\varepsilon[1] = (\delta^{m_1}, \delta^{m_1}, \delta^{m_2}, \delta^{m_2}, \delta^{m_3}, \delta^{m_3}),$$

where  $(m_1, m_2, m_3)$  is (odd, even, odd) (resp. (even, odd, odd)) for the first (resp. second) case. By these conditions, the cluster variables  $x[k]_j$  are solved up to constant. We can see from a three-dimensional interpretation of flips in Fig. 11 that the cluster variable  $x[k]_j$  is assigned to an edge of ideal tetrahedra  $\triangle_i(F_k)$ , and (4.19) and (4.21) support that identical edges have same complex numbers. Then the cluster variables are identified with Zickert's complex variables on edges, and the complex volume can be computed from the cluster variables.

To compute the complex volume modulo  $\pi^2$ , we should fix a vertex ordering of each ideal tetrahedra  $\triangle_i(F_k)$ . Although, contrary to the once-punctured torus bundle, an orientation of triangulations of  $\Sigma_{0,4}$  cannot be fixed uniquely due to the transformation group  $\Gamma$ , and it is tedious to give orientations from  $q/p$ . So for simplicity, we discard vertex ordering, and compute the complex volume modulo  $\frac{\pi^2}{6}$ . From (4.11) and (4.17), we obtain the following.

**Lemma 4.6.** *Let  $(x[k], \varepsilon[k])$  be a cluster pattern which satisfies the conditions (4.18)–(4.21). Then the moduli  $z_1[k]$  and  $z_2[k]$  of a pair of tetrahedra  $\triangle_1(F_k)$  and  $\triangle_2(F_k)$  are given from the cluster pattern,*

$$z_i[k] = \begin{cases} -\frac{1}{\varepsilon[k]_i} \frac{x[k]_5 x[k]_6}{x[k]_3 x[k]_4}, & \text{for } F_k = R, \\ -\frac{1}{\varepsilon[k]_{2+i}} \frac{x[k]_1 x[k]_2}{x[k]_5 x[k]_6}, & \text{for } F_k = L. \end{cases} \quad (4.22)$$

for  $k = 1, \dots, c-2$  and  $i = 1, 2$ .

Then we obtain the flattenings of the tetrahedra  $\triangle_i(F_k)$  as follows.

**Lemma 4.7.** *We follow the setting of Lemma 4.6. The flattening  $(z_i[k]; p_i[k], q_i[k])$  for a tetrahedron  $\triangle_i(F_k)$  are given by*

$$\begin{aligned} \log z_i[k] + p_i[k] \pi i &= \begin{cases} \log x[k]_5 + \log x[k]_6 - \log x[k]_3 - \log x[k]_4, & \text{for } F_k = R, \\ \log x[k]_1 + \log x[k]_2 - \log x[k]_5 - \log x[k]_6, & \text{for } F_k = L, \end{cases} \\ & - \log(1 - z_i[k]) + q_i[k] \pi i \\ &= \begin{cases} \log x[k]_3 + \log x[k]_4 - \log x[k]_i - \log x[k+1]_{7-i}, & \text{for } F_k = R. \\ \log x[k]_5 + \log x[k]_6 - \log x[k]_{2+i} - \log x[k+1]_{7-i}, & \text{for } F_k = L, \end{cases} \end{aligned} \quad (4.23)$$

for  $k = 1 \dots, c-2$  and  $i = 1, 2$ .

*Proof.* Due to explicit forms of flips (4.15), we have

$$\frac{1}{1 - z_i[k]} = \begin{cases} \frac{\varepsilon[k]_i}{1 \oplus \varepsilon[k]_i} \frac{x[k]_3 x[k]_4}{x[k]_i x[k+1]_{7-i}}, & \text{for } F_k = R, \\ \frac{\varepsilon[k]_{2+i}}{1 \oplus \varepsilon[k]_{2+i}} \frac{x[k]_5 x[k]_6}{x[k]_{2+i} x[k+1]_{7-i}} & \text{for } F_k = L, \end{cases} \quad (4.24)$$

for  $i = 1, 2$ . By comparing (4.22), (4.24) and a form of (2.15), we get the claim.  $\square$

**Theorem 4.8.** *The complex volume of  $S^3 \setminus K_{q/p}$  is given by the flattenings of Lemma 4.7 as*

$$i \left( \text{Vol}(S^3 \setminus K_{q/p}) + i \text{CS}(S^3 \setminus K_{q/p}) \right) = \sum_{k=1}^{c-3} \sum_{i=1,2} L(z_i[k]; p_i[k], q_i[k]) \mod \frac{\pi^2}{6}. \quad (4.25)$$

*Remark 4.9.* Vertex ordering of each tetrahedron can be determined case by case. As an example, we study  $K_{[a+1,2]}$ , which has a cluster pattern

$$(\mathbf{x}[1], \varepsilon[1]) \xrightarrow{R} (\mathbf{x}[2], \varepsilon[2]) \xrightarrow{R} \cdots \xrightarrow{R} (\mathbf{x}[a+1], \varepsilon[a+1]).$$

Based on triangulations of knot complements, we may choose vertex ordering so that

$$(\text{sgn}(\Delta_1(F_k)), \text{sgn}(\Delta_2(F_k))) = \begin{cases} (-1, -1), & \text{for odd } k \text{ and } k \neq a, a-1, \\ (+1, +1), & \text{for even } k \text{ and } k \neq a, \\ (+1, -1), & \text{for odd } k \text{ and } k = a, a-1, \\ (-1, -1), & \text{for even } k \text{ and } k = a. \end{cases}$$

We can then identify

$$\begin{aligned} & (z_1[k], z_2[k]) \\ &= \begin{cases} \left( \frac{1 \oplus \varepsilon[k]_1}{\varepsilon[k]_1} \frac{x[k]_1 x[k+1]_6}{x[k]_3 x[k]_4}, \frac{1 \oplus \varepsilon[k]_2}{\varepsilon[k]_2} \frac{x[k]_2 x[k+1]_5}{x[k]_3 x[k]_4} \right), & \text{for odd } k \text{ and } k \neq a, a-1, \\ \left( -\frac{1}{\varepsilon[k]_1} \frac{x[k]_5 x[k]_6}{x[k]_3 x[k]_4}, -\frac{1}{\varepsilon[k]_2} \frac{x[k]_5 x[k]_6}{x[k]_3 x[k]_4} \right), & \text{for even } k \text{ and } k \neq a, \\ \left( (1 \oplus \varepsilon[k]_1) \frac{x[k]_1 x[k+1]_6}{x[k]_5 x[k]_6}, \frac{1 \oplus \varepsilon[k]_2}{\varepsilon[k]_2} \frac{x[k]_2 x[k+1]_5}{x[k]_3 x[k]_4} \right), & \text{for odd } k \text{ and } k = a, a-1, \\ \left( \frac{1}{1 \oplus \varepsilon[k]_1} \frac{x[k]_5 x[k]_6}{x[k]_1 x[k+1]_6}, \frac{1}{1 \oplus \varepsilon[k]_2} \frac{x[k]_5 x[k]_6}{x[k]_2 x[k+1]_5} \right), & \text{for even } k \text{ and } k = a. \end{cases} \end{aligned} \quad (4.26)$$

Accordingly, we have

$$\begin{aligned}
 & \left( \frac{1}{1 - z_1[k]}, \frac{1}{1 - z_2[k]} \right) \\
 = & \begin{cases} \left( -\varepsilon[k]_1 \frac{x[k]_3 x[k]_4}{x[k]_5 x[k]_6}, -\varepsilon[k]_2 \frac{x[k]_3 x[k]_4}{x[k]_5 x[k]_6} \right), & \text{for odd } k \text{ and } k \neq a, a-1, \\ \left( \frac{\varepsilon[k]_1}{1 \oplus \varepsilon[k]_1} \frac{x[k]_3 x[k]_4}{x[k]_1 x[k+1]_6}, \frac{\varepsilon[k]_2}{1 \oplus \varepsilon[k]_2} \frac{x[k]_3 x[k]_4}{x[k]_2 x[k+1]_5} \right), & \text{for even } k \text{ and } k \neq a, \\ \left( -\frac{1}{\varepsilon[k]_1} \frac{x[k]_5 x[k]_6}{x[k]_3 x[k]_4}, -\varepsilon[k]_2 \frac{x[k]_3 x[k]_4}{x[k]_5 x[k]_6} \right), & \text{for odd } k \text{ and } k = a, a-1, \\ \left( \frac{1 \oplus \varepsilon[k]_1}{\varepsilon[k]_1} \frac{x[k]_1 x[k+1]_6}{x[k]_3 x[k]_4}, \frac{1 \oplus \varepsilon[k]_2}{\varepsilon[k]_2} \frac{x[k]_2 x[k+1]_5}{x[k]_3 x[k]_4} \right), & \text{for even } k \text{ and } k = a. \end{cases}
 \end{aligned} \tag{4.27}$$

These data give the flattening  $(z_i[k]; p_i[k], q_i[k])$  for each tetrahedra from (2.15), and we get

$$\begin{aligned}
 & i \left( \text{Vol}(S^3 \setminus K_{[a+1,2]}) + i \text{CS}(S^3 \setminus K_{[a+1,2]}) \right) \\
 & = \sum_{k=1}^a \sum_{i=1,2} \text{sgn}(\triangle_i(F_k)) L(z_i[k]; p_i[k], q_i[k]) \mod \pi^2.
 \end{aligned} \tag{4.28}$$

#### 4.5 Example: $6_1$

The knot  $6_1$  is a two-bridge knot  $K_{[4,2]}$ , which corresponds to  $a = 3$  in Remark 4.9. We set an initial seed as

$$\mathbf{x}[1] = (1, 1, 1, 1, x, x), \quad \boldsymbol{\varepsilon}[1] = (\delta^0, \delta^0, \delta^1, \delta^1, \delta^1, \delta^1),$$

and a constraint (4.21) corresponds to

$$x(2 + x^2) = 1 + 3x^2 + x^4.$$

A geometric solution is  $x = 0.1048 \dots - i \cdot 1.5524 \dots$ , and the flattenings are calculated from (4.26) and (4.27) to be

$k$	$(z_1[k], z_2[k])$	$(p_1[k], p_2[k])$	$(q_1[k], q_2[k])$
1	$(-1.3992 \dots - i \cdot 0.3256 \dots, -1.3992 \dots - i \cdot 0.3256 \dots)$	$(0, 0)$	$(1, 1)$
2	$(1.8518 \dots + i \cdot 0.9112 \dots, 1.8518 \dots + i \cdot 0.9112 \dots)$	$(-2, -2)$	$(-1, -1)$
3	$(0.8951 \dots + i \cdot 1.552 \dots, 0.9566 \dots - i \cdot 0.6412 \dots)$	$(-2, 0)$	$(1, -1)$

We get from (4.28)

$$\text{Vol}(S^3 \setminus 6_1) + i \text{CS}(S^3 \setminus 6_1) = 3.1639 \dots + i \cdot 3.0788 \dots \in \mathbb{C}/i\pi^2 \mathbb{Z}.$$

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